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Further Results on Entire Functions That Share One Value with Their Derivatives*

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This paper deals with problems of the uniqueness of entire functions that share one value with their derivatives. The results in this paper generalize a result of Jank, Mues, and Volkmann and answer a question posed by H. Zhong and a question of Yi and Yang.

1. INTRODUCTION

Let f and g be two nonconstant meromorphic functions in the complex plane. We say two meromorphic functions f and g share a finite value a *IM* (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we say that f and g share the value of *CM* (counting multiplicities). It is assumed that the reader is familiar with the basic notations and the fundamental results of Nevanlinna's theory of meromorphic functions, as found in [1].

In 1977, L. Rubel and C. C. Yang proved (see [2])

THEOREM A. *Let f be a nonconstant entire function. If f and f' share two finite, distinct values a and b CM, then $f = f'$.*

In 1986, Jank, Mues, and Volkmann proved in [3]

THEOREM B. *Let f be a nonconstant entire function. If f and f' share the value a $(a \neq 0, \infty)$ IM and $f'' = a$ when $f = a$, then $f \equiv f'$.*

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Now let f and g be two nonconstant meromorphic functions and let $a \in C \setminus \{0\}$ be fixed. Let $E = E(f, g, a, a) = \{f^{-1}(a) \cup g^{-1}(a)\} \setminus \{f^{-1}(a) \cap g^{-1}(a)\}$ and let $N_E(r, 1/(f-a))$ denote the counting function of a a-point of f which belongs to the set E ($N_E(r, 1/(g-a))$ is defined similarly). We say f and g share the value a *IMN* if f and g share the value a *IM* outside the set E, where $E = E(f, g, a, a)$ satisfies that $N_E(r, 1/(f-a)) = S(r, f)$ and $N_E(r, 1/(g-a)) = S(r, g)$ (see [4]). In 1995, Hauliang Zhong proved [4]:

THEOREM C. *Let f be a nonconstant entire function. Suppose f and $f^{(n)}$ $(n \geq 1)$ share the value $a(\neq 0, \infty)$ IMN and $f' = f^{(n+1)} = a$ when $f = a$. If for any set e of finite linear measure*

$$\lim_{\substack{r \to \infty \\ r \notin e}} \frac{N_E\left(r, 1/\left(f^{(n+1)} - a\right)\right)}{N\left(r, 1/\left(f^{(n+1)} - a\right)\right)} \neq \frac{1}{2}, \tag{1}$$

where $E = E(f, f^{(n+1)}, a, a)$, then $f \equiv f^{(n)}$.

Theorems C suggests the following question:

QUESTION 1. *Is it possible to get rid of condition* (1) *in Theorem C*? (See [4, p.251]).

In this paper, we will give a positive answer to Question 1 and discuss a question of Yi and Yang (mentioned in Section 4).

2. SOME LEMMAS

LEMMA 1. *Let f be a nonconstant meromorphic function and a $(\neq 0)$ be a finite complex number. Then for any positive integer n,*

$$m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f^{(n)} - a}\right) \leq m\left(r, \frac{1}{f^{(n+1)}}\right) + S(r, f).$$

Proof. This can be derived simply from Lemma 1 of [4].

LEMMA 2. *Let f be a nonconstant entire function. If f and $f^{(n)}$ share the value a $(\neq 0, \infty)$ IMN and $f' = a$, $f^{(n+1)} = a$ when $f = a$, then*

$$T(r, f) = 2N\left(r, \frac{1}{f-a}\right) + S(r, f).$$

Proof. By Lemma 1, we have

$$\begin{aligned}
&T(r,f) + T(r,f^{(n)}) \\
&\quad \le N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f^{(n)}-a}\right) + m\left(r, \frac{1}{f^{(n+1)}}\right) + S(r,f) \\
&\quad \le 2N\left(r, \frac{1}{f-a}\right) + T(r, f^{(n+1)}) + S(r,f).
\end{aligned}$$

Since

$$\begin{aligned}
T(r, f^{(n+1)}) = m(r, f^{(n+1)}) &\le m(r, f^{(n)}) + S(r,f) \\
&= T(r, f^{(n)}) + S(r,f),
\end{aligned}$$

we get

$$T(r,f) \le 2N\left(r, \frac{1}{f-a}\right) + S(r,f).$$

On the other hand

$$2N\left(r, \frac{1}{f-a}\right) \le N\left(r, \frac{1}{f-f^{(n)}}\right) + S(r,f) \le T(r,f) + S(r,f).$$

Lemma 2 is thus proved.

LEMMA 3. *Let f be a nonconstant entire function. If the conditions of Lemma 2 are satisfied, then*

$$T(r,f) = T(r, f^{(n)}) + S(r,f).$$

Proof. If $f^{(n)} \equiv f^{(n+1)}$, then $f = ce^z + P(z)$ for a polynomial $P(z)$ and the lemma holds. Now we suppose that $f^{(n)} \not\equiv f^{(n+1)}$ and derive from the conditions that

$$\begin{aligned}
N\left(r, \frac{1}{f-a}\right) &\le N\left(r, \frac{1}{f^{(n+1)}/f^{(n)} - 1}\right) + S(r,f) \\
&\le T\left(r, \frac{f^{(n+1)}}{f^{(n)}}\right) + S(r,f) \\
&\le N\left(r, \frac{1}{f^{(n)}}\right) + S(r,f). \qquad (2)
\end{aligned}$$

From Lemma 2, we deduce that

$$T(r,f) = 2m\left(r, \frac{1}{f-a}\right) + S(r,f)$$
$$\leq 2m\left(r, \frac{1}{f^{(n)}}\right) + S(r,f),$$

and so

$$2N\left(r, \frac{1}{f^{(n)}}\right) \leq 2T\left(r, f^{(n)}\right) - T(r,f) + S(r,f)$$
$$\leq T\left(r, f^{(n)}\right) + S(r,f).$$

Combining with (2), we have

$$T\left(r, f^{(n)}\right) \leq T(r,f) + S(r,f)$$
$$= 2N\left(r, \frac{1}{f-a}\right) + S(r,f)$$
$$\leq T\left(r, f^{(n)}\right) + S(r,f).$$

Lemma 3 is thus proved.

LEMMA 4 [5]. *Let f be a nonconstant meromorphic function and let*

$$P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n,$$

where a_j $(j = 0, 1, 2, \ldots, n)$ are constants and $a_0 \neq 0$. Then

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

LEMMA 5 [6]. *Let f_j $(j = 1, 2, \ldots, m-1)$ $(m \geq 3)$ be nonconstant meromorphic functions and $f_m \not\equiv 0$ be meromorphic in the complex plane. If there exists $\lambda < 1$ such that*

(i) $\sum_{j=1}^{m} f_j(z) = 1$, *and*

(ii) $\sum_{j=1}^{m} N(r, 1/f_j) + (m-1)\sum_{j=1}^{m} N(r, \; f_j) \leq (\lambda + o(1))T(r, f_k)$ $(k = 1, 2, \ldots, m-1)$, *then* $f_m \equiv 1$.

3. THEOREMS AND THEIR PROOFS

THEOREM 1. *Let f be a nonconstant entire function. If f and $f^{(n)}$ $(n \geq 1)$ share the value $a(\neq 0, \infty)$ IMN, and $f' = a$, $f^{(n+1)} = a$ when $f = a$, then $f \equiv ce^z$, where $c \in C \setminus \{0\}$.*

Proof. We set

$$\alpha = \frac{f^{(n+1)} - f'}{f-a}, \qquad \beta = f^{(n+1)}\left(\frac{1}{f-a} - \frac{1}{f^{(n)}-a}\right) = \frac{f^{(n+1)}\left(f^{(n)} - f\right)}{(f-a)\left(f^{(n)}-a\right)}.$$

Then it is clear that

$$m(r,\alpha) = S(r,f), \qquad m(r,\beta) = S(r,f). \tag{3}$$

Now suppose z_1 is an a-point of $f^{(n)}$ and f. Since $f'(z_1) = f^{(n+1)}(z_1) \neq 0$, we know that z_1 is a simple zero of $f - a$ and $f^{(n)} - a$. Therefore

$$N(r,\alpha) = 0, \qquad N(r,\beta) = S(r,f)$$

and we conclude that

$$T(r,\alpha) + T(r,\beta) = S(r,f).$$

From the Taylor expansion of f at the point z_1 mentioned above, it is easy to compute that

$$\alpha(z_1) = \frac{f^{(n+2)}(z_1) - f''(z_1)}{a}, \qquad \beta(z_1) = \frac{f^{(n+2)}(z_1) - f''(z_1)}{2a}. \tag{4}$$

If $2\beta \not\equiv \alpha$, by Lemma 2 and (4), we have

$$\begin{aligned} T(r,f) &= 2N\left(r, \frac{1}{f-a}\right) + S(r,f) \\ &\leq 2N\left(r, \frac{1}{2\beta - \alpha}\right) + S(r,f) \\ &\leq 2T(r, 2\beta - \alpha) + S(r,f) = S(r,f), \end{aligned}$$

which is a contradiction. Thus we conclude $2\beta \equiv \alpha$, and so

$$\left(f^{(n)} - a\right)\left(f^{(n+1)} - f'\right) \equiv 2f^{(n+1)}\left(f^{(n)} - f\right).$$

Hence

$$\frac{f^{(n)} - f}{\left(f^{(n)} - a\right)^2} \equiv c(\text{constant}). \tag{5}$$

If $c \neq 0$, then from (5) we have

$$f = f^{(n)} - c\left(f^{(n)} - a\right)^2,$$

and by Lemma 4

$$T(r,f) = 2T\big(r,f^{(n)}\big) + S(r,f),$$

which contradicts Lemma 3. Therefore $c = 0$ and $f \equiv f^{(n)}$.

Solving the equation $f^{(n)} = f$, we have

$$f = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \cdots + c_n e^{\lambda_n z}, \tag{6}$$

where $\lambda_1 = 1$ and $\lambda_i \neq 1$ $(i = 2, 3, \ldots, n)$ are the distinct roots of $\lambda^n - 1 = 0$ and c_j $(j = 1, 2, \ldots, n)$ are constants.

Set

$$\phi = \frac{f - f'}{f - a} \equiv \frac{f^{(n)} - f'}{f - a}. \tag{7}$$

Then from (6) and (7), we have

$$T(r,\phi) = S(r,f) = S(r,e^z).$$

We claim that $\phi \equiv 0$. Certainly, this claim is sufficient to establish Theorem 1. Thus, we assume to the contrary that $\phi \not\equiv 0$ and seek a contradiction. From (6) and (7), we have $n \geq 2$ and

$$\frac{c_1}{a} e^{\lambda_1 z} + \frac{\phi + 1 - \lambda_2}{a\phi} c_2 e^{\lambda_2 z} + \cdots + \frac{\phi + 1 - \lambda_n}{a\phi} c_n e^{\lambda_n z} \equiv 1. \tag{8}$$

We denote by γ_k the coefficient of $e^{\lambda_k z}$ $(1 \leq k \leq n)$ in (8) and we denote by γ_{k_j} $(j = 1, 2, \ldots, m;\ 2 \leq m \leq n)$ the nonzero coefficients in $\{\gamma_k\}_{k=1}^{k=n}$.

Let $f_j = \gamma_{k_j} e^{\lambda_{k_j} z}$, $j = 1, 2, \ldots, m$. It is clear that the conditions of Lemma 5 are satisfied. If $m \geq 3$, by Lemma 5, $f_m \equiv 1$, which is impossible. If $m = 2$, then

$$\gamma_{k_1} e^{\lambda_{k_1} z} + \gamma_{k_2} e^{\lambda_{k_2} z} \equiv 1.$$

By the second fundamental theorem, we have

$$\begin{aligned} T(r,e^z) = T(r, e^{\lambda_{k_1} z}) &\leq N\left(r, \frac{1}{e^{\lambda_{k_1} z} - 1/\gamma_{k_1}}\right) + S(r,e^z) \\ &= N\left(r, \frac{\gamma_{k_1}}{\gamma_{k_2}}\right) + S(r,e^z) \\ &= S(r,e^z). \end{aligned}$$

This is also impossible. The proof of Theorem 1 is complete.

THEOREM 2. *Let f by an entire function that satisfies the following conditions*:

(i) *f and f' share the finite value $a \neq 0$ IM, and $f^{(n)} = a$ $(n \geq 2)$ when $f = a$ and*

(ii) $\delta(0, f) > 0$. *Then $f = ce^z$ and so $f \equiv f'$.*

Proof. Set

$$\varphi = \frac{f^{(n)} - f'}{f - a}.$$

Thus $m(r, \varphi) = S(r, f)$. Since $a \neq 0$ is a shared value of f and f', we know that the zeros of $f - a$ are simple zeros and so φ is an entire function satisfying

$$T(r, \varphi) = S(r, f).$$

If $\varphi \not\equiv 0$, then

$$\frac{1}{f} = \frac{1}{a\varphi}\left(\varphi - \frac{f^{(n)}}{f} + \frac{f'}{f}\right), \qquad \text{and} \qquad m\left(r, \frac{1}{f}\right) = S(r, f).$$

Since $\delta(0, f) > 0$, we have a contradiction. If $\varphi \equiv 0$, we have $f' \equiv f^{(n)}$, and $f = f^{(n-1)} + d$, where d is a constant. If $d \neq 0$, then

$$m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{d}\left(1 - \frac{f^{(n-1)}}{f}\right)\right) = S(r, f).$$

This is also a contradiction, and thus we get $f \equiv f^{(n-1)}$. If $n = 2$, Theorem 2 holds. If $n \geq 3$, then f and f' share the value a *IM* and $f^{(n-1)} = a$ when $f(z) = a$. By using the same methods, we get $f \equiv f^{(n-2)}$. By applying the same methods $n - 1$ times, it follows that $f \equiv f'$ and $f = ce^z$. Theorem 2 is thus proved.

Remark. Theorem 1 gives a positive answer to Question 1. Theorem 2 gives a generalization of Theorem B with the additional condition that $\delta(0,f) > 0$. The example $f = e^{az} + a - 1$ shows that the condition $\delta(0, f) > 0$ in Theorem 2 is essential, where $a \neq 1$ is a $(n - 1)$st $(n \geq 3)$ root of unity. The condition $a \neq 0$ in Theorems 1 and 2 is indispensable in view of a counterexample in [3].

4. A QUESTION OF YI AND YANG

Recently, Yi and Yang posed the following question in [6]

QUESTION 2. *Let $f(z)$ be a nonconstant meromorphic function and $a \neq 0$ a finite complex number. If f, $f^{(n)}$, and $f^{(m)}$ share the value a CM, where n and m $(n < m)$ are distinct positive integers not all even or add, must $f \equiv f^{(n)}$?*

Let n and m be the integers mentioned above, $\omega \neq 1$ a root of the equation $\omega^{m-n} = 1$ with $\omega^n = \omega^m \neq 1$. Set $a = \omega^n$ and $f(z) = e^{az} + a - 1$. It is clear that f, $f^{(n)}$, and $f^{(m)}$ share the value a *CM* and $f \neq f^{(n)}$. This example says that the answer to Question 2 is negative. Similarly to the proof of Theorem 2, we will have a positive answer if f satisfies the additional condition $\delta(0, f) > 0$.

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